17 Applications of Differentiation I

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1. [13 marks: 3, 2, 4, 4] [TISC]

(a) The expression $\dfrac{x^3+1}{x^2-1}$ can be rewritten as $px + \dfrac{qx+r}{x^2-1}$. Find p, q and r.

(b) State the equations of all the asymptotes of the curve $y = \dfrac{x^3+1}{x^2-1}$

(c) The curve with equation $y = \dfrac{x^3+1}{x^2-1}$ has a maximum point at (0, –1).

Use Calculus to show that the curve has a local minimum point at (2, 3).

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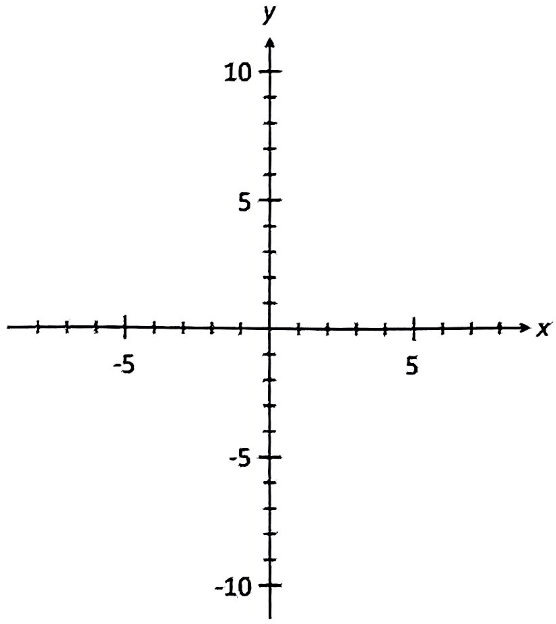
1. (d) Sketch the graph of $y = \dfrac{x^3+1}{x^2-1}$. Indicate clearly the intercepts, stationary points, asymptotes and any other important features.



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2. [14 marks: 1, 1, 4, 4, 4] [TISC]

Consider the curve with equation $y^2 = \dfrac{x^4}{x^2 - 1}$.

(a) Find the equation of the vertical asymptote(s).

(b) Find the x-intercept(s) and the y-intercept(s) of the curve.

(c) Determine $\lim\limits_{x \to \pm\infty} y$. Hence, find the equation of the oblique asymptotes.

(d) Use differentiation to verify that $(\sqrt{2}, 2)$ is a minimum point on this curve.

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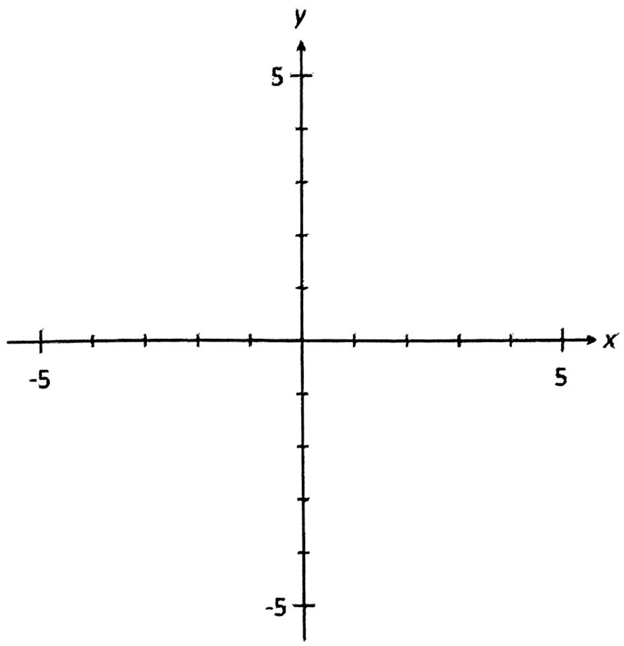
2. (e) Sketch the curve $y^2 = \dfrac{x^4}{x^2 - 1}$.

Indicate clearly all the important features of this curve.



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3. [12 marks: 4, 4, 4] [TISC]

Consider the curve with equation $y = \dfrac{x^3}{x^2 - 4}$.

(a) Find the equation of the asymptote(s).

(b) Use differentiation to find the number of stationary points on this curve.

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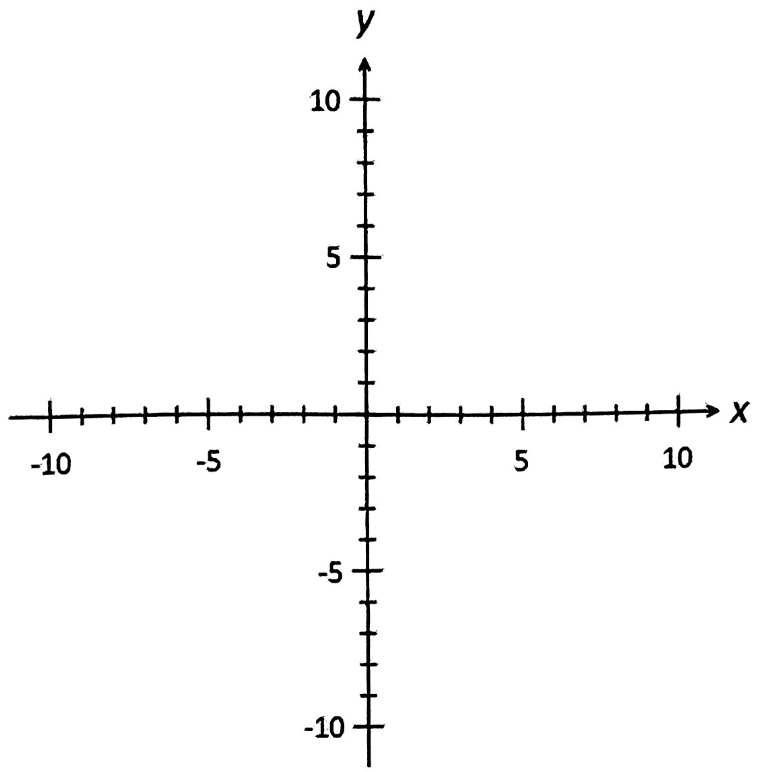
3. (c) Sketch this curve on the axes below.
Indicate clearly all intercepts, stationary points and asymptotes.



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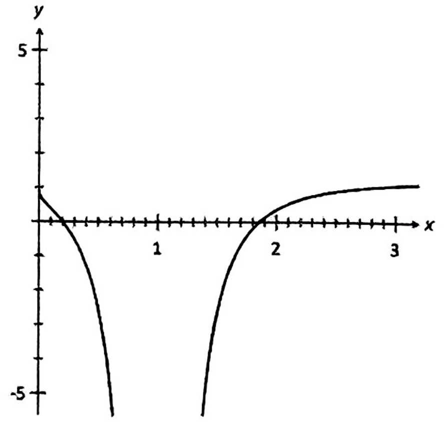
4. [7 marks: 4, 3]

Consider the curve with equation $y = x + \dfrac{x}{\ln x} - 10$.

(a) Use an analytical method to show that this curve has turning points when $(\ln x)^2 + \ln x - 1 = 0$.

(b) Use the graph of $y = 1 + \dfrac{\ln x - 1}{(\ln x)^2}$ shown below, to determine correct to one decimal place, the <u>x-coordinate</u> of the maximum point of this curve. Use either the sign test or the second derivative test to identify the maximum turning point.



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8. (b) Find the equation of the line passing through the point (1, 1) and parallel to the tangent to this curve at the point ($ln\,2$, 1).

> Gradient $m = \dfrac{2}{1-2\ln 2}$. ✓
> Hence, equation of tangent is
> $y - 1 = \dfrac{2}{1-2\ln 2}(x-1)$ ✓
> $y = \dfrac{2x}{1-2\ln 2} - \left(\dfrac{1+2\ln 2}{1-2\ln 2}\right)$ ✓

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9. [8 marks: 5, 3] [TISC]

A curve has equation curve $x^2y + \sqrt{3+y^2} = 3$.

(a) Find the equation of the tangent to this curve at the point (–1, 1).

> $2xy + x^2\dfrac{dy}{dx} + \dfrac{1}{2}(3+y^2)^{-\frac{1}{2}}\left(2y\dfrac{dy}{dx}\right) = 0$ ✓ ✓ ✓
> Subst. $x = -1, y = 1$
> $-2 + \dfrac{dy}{dx} + \dfrac{1}{2}\dfrac{dy}{dx} = 0$
> $\dfrac{dy}{dx} = \dfrac{4}{3}$ ✓
> Equation of tangent is:
> $y - 1 = \dfrac{4}{3}(x+1)$
> $y = \dfrac{4x}{3} + \dfrac{7}{3}$ ✓

(b) Use the method of incremental change to find the change in y when x changes from –1.00 to –1.01.

> $\delta y \approx \left.\dfrac{dy}{dx}\right|_{x=-1,y=1} \times \delta x$ ✓
> $= \dfrac{4}{3} \times (-0.01) \approx -0.013$ ✓✓

17 Applications of Differentiation I

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1. [13 marks: 3, 2, 4, 4] [TISC]

(a) The expression $\dfrac{x^3+1}{x^2-1}$ can be rewritten as $px + \dfrac{qx+r}{x^2-1}$. Find p, q and r.

> $x^3 + 1 = px(x^2-1) + qx + r$
> Compare x^3 coefficient: $p = 1$ ✓
> Subst. $x = 0$: $r = 1$ ✓
> Subst. $x = 1$: $q + r = 2$
> $\Rightarrow$ $q = 1$ ✓

(b) State the equations of all the asymptotes of the curve $y = \dfrac{x^3+1}{x^2-1}$

> Vertical asymptote: $x = 1$ ✓
> Oblique asymptote: $y = x$ ✓

(c) The curve with equation $y = \dfrac{x^3+1}{x^2-1}$ has a maximum point at (0, –1).
Use Calculus to show that the curve has a local minimum point at (2, 3).

> $\dfrac{dy}{dx} = \dfrac{3x^2(x^2-1)-(x^3+1)(2x)}{(x^2-1)^2}$ ✓
> When $x = 2$, $\dfrac{dy}{dx} = 0$. ✓

x	2^-	2	2^+
dy/dx	–	0	+

> ✓
> Hence, (2, 3) is a minimum point. ✓

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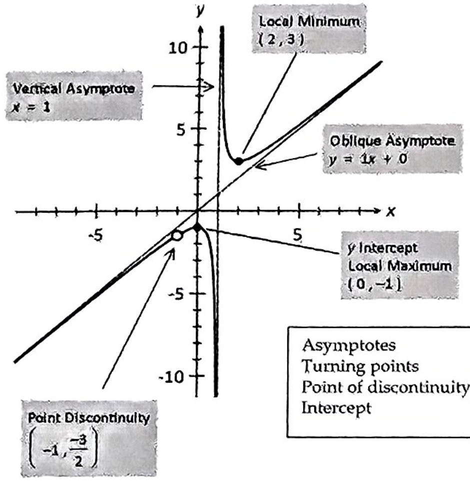
1. (d) Sketch the graph of $y = \dfrac{x^3+1}{x^2-1}$. Indicate clearly the intercepts, stationary points, asymptotes and any other important features.



Asymptotes	✓
Turning points	✓
Point of discontinuity	✓
Intercept	✓

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2. [14 marks: 1, 1, 4, 4, 4] [TISC]

Consider the curve with equation $y^2 = \dfrac{x^4}{x^2-1}$.

(a) Find the equation of the vertical asymptote(s).

$x = -1, x = 1$ ✓

(b) Find the x-intercept(s) and the y-intercept(s) of the curve.

(0, 0) ✓

(c) Determine $\lim\limits_{x \to \pm\infty} y$. Hence, find the equation of the oblique asymptotes.

Using polynomial division:

$$y^2 = x^2 + 1 + \frac{1}{x^2-1}$$ ✓✓

$$y = \pm\sqrt{\left(x^2+1+\frac{1}{x^2-1}\right)}$$ ✓

Hence:

$\lim\limits_{x\to\infty} y = \pm x$ and $\lim\limits_{x\to-\infty} y = \pm x$ ✓

(d) Use differentiation to verify that (√2, 2) is a minimum point on this curve.

$$2y\frac{dy}{dx} = \frac{4x^3(x^2-1)-x^4(2x)}{(x^2-1)^2}$$ ✓

When $x = \sqrt{2}$, $y = 2$: $4 \times \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = 0.$ ✓

x	$\sqrt{2}^-$	$\sqrt{2}$	$\sqrt{2}^+$
dy/dx	−	0	+

✓

Hence, (√2, 2) is a minimum point. ✓

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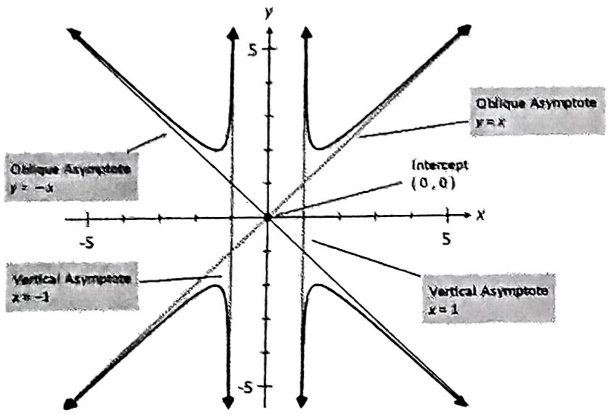
2. (e) Sketch the curve $y^2 = \dfrac{x^3}{x^2 - 1}$.

Indicate clearly all the important features of this curve.



Point (0, 0)	✓
Asymptotes	✓
Symmetrical about x-axis	✓
Symmetrical about y-axis	✓

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3. [12 marks: 4, 4, 4] [TISC]

Consider the curve with equation $y = \dfrac{x^3}{x^2 - 4}$.

(a) Find the equation of the asymptote(s).

Vertical Asymptotes: $x = -2, x = 2$ ✓✓

By polynomial division: $y = x + \dfrac{4x}{x^2 - 4}$. ✓

Hence, oblique asymptote: $y = x$ ✓

(b) Use differentiation to find the number of stationary points on this curve.

$\dfrac{dy}{dx} = \dfrac{(x^2-4)(3x^2) - x^3(2x)}{(x^2-4)^2}$ ✓

$\dfrac{dy}{dx} = 0 \Rightarrow 3x^4 - 12x^2 - 2x^4 = 0$ ✓

$x^2(x^2 - 12) = 0$

$x = 0, \pm\sqrt{12}$ ✓

$y = \dfrac{x^3}{x^2 - 4}$ is defined for $x = 0$ and $\pm\sqrt{12}$.

Hence, there are three stationary points. ✓

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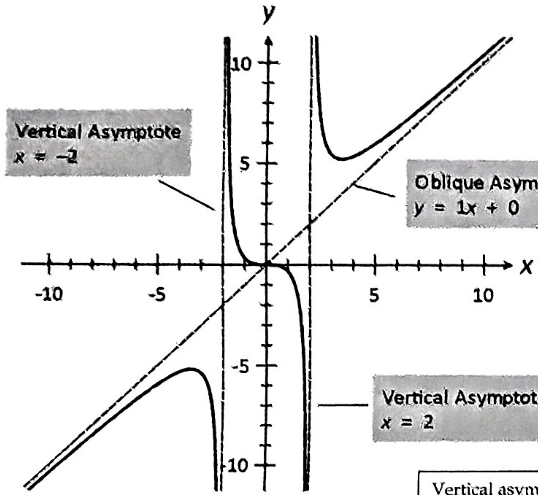
3. (c) Sketch this curve on the axes below.
Indicate clearly all intercepts, stationary points and asymptotes.



Vertical asymptotes	✓
Oblique asymptotes	✓
All correct	✓✓

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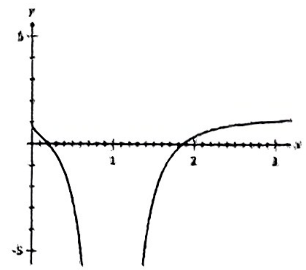
4. [7 marks: 4, 3]

Consider the curve with equation $y = x + \frac{x}{\ln x} - 10$.

(a) Use an analytical method to show that this curve has turning points when $(\ln x)^2 + \ln x - 1 = 0$.

> $\frac{dy}{dx} = 1 + \frac{\ln x - 1}{(\ln x)^2}$ ✓✓
>
> $\frac{dy}{dx} = 0 \Rightarrow \quad \ln x - 1 = -(\ln x)^2$ ✓
>
> $(\ln x)^2 + \ln x - 1 = 0$ ✓

(b) Use the graph of $y = 1 + \frac{\ln x - 1}{(\ln x)^2}$ shown below, to determine correct to one decimal place, the x-coordinate of the maximum point of this curve. Use either the sign test or the second derivative test to identify the maximum turning point.



> $\frac{dy}{dx} = 1 + \frac{\ln x - 1}{(\ln x)^2}$.
>
> From graph of $y = 1 + \frac{\ln x - 1}{(\ln x)^2}$, roots are $x \approx 0.2, 1.9$.
>
> Hence, $\frac{dy}{dx} = 0$ when $x \approx 0.2, 1.9$. ✓
>
> From graph, for $x < 0.2$, $\frac{dy}{dx} > 0$ and for $x > 0.2$, $\frac{dy}{dx} < 0$. ✓
>
> Hence, there is a maximum point at $x \approx 0.2$. ✓